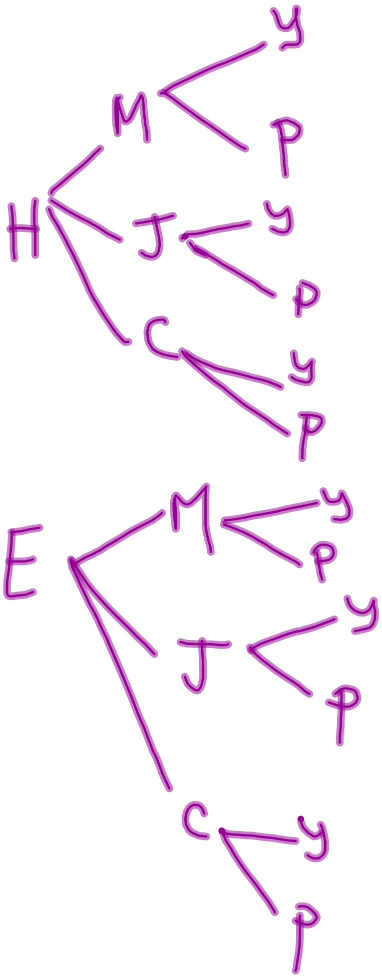
Chapter 2: Counting Methods

Section 2.1: Counting Principles

Example 1: A cafe has a lunch special consisting of one from each category.

- ham sandwich, egg sandwich 2
- milk, juice, coffee 3
- yogurt, pie 2

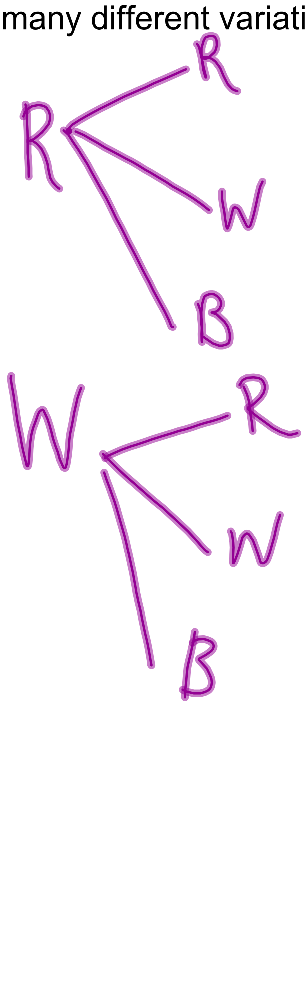
In how many ways can you choose your lunch? Use a tree diagram.

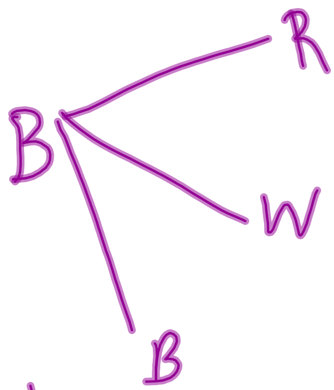


12 ways to choose lunch

Example 2: Hannah plays on her school soccer team. The team uniform has:
3 different sweaters (red, white, black)
3 different shorts: (red, white, black)

How many different variations are there?





9 variations

Fundamental Counting Principle:

- helps determine the total # of possible arrangements that can occur within a group(s)
- applies only when tasks are related by the word "and" (never "or")
- if one task can be performed in 'a' ways and another task can be performed in 'b' ways, then both tasks can be performed in a x b ways

Example 3: A luggage lock has a three digit code that uses the digits 0 - 9.
A) How many different three digit codes are possible?

$\underline{10}\ \underline{10}\ \underline{10} = 1000$

B) Suppose each digit can only be used once. How many different three digit codes are possible?

$\underline{10}\ \underline{9}\ \underline{8} = 720$

Complete questions pg 73 - 75

#1(use tree diagram for part (a)), 2, 3(very important!!), 4 - 11,14 - 16

Section 2.2: Introducing Permuations and Factorial Notation

Factorial Notation: n! = n(n-1)(n-2)....(3)(2)(1)

Note:
- factorial notation is only defined for natural numbers therefore expressions like (-2)! and (1\2)! have no meaning
- $\boxed{0! = 1}$
-

Example: 4! = 4 x 3 x 2 x 1
= 24

Example 1: Evaluate

5!

$= 5 \times 4 \times 3 \times 2 \times 1$

$= 120$

$$\frac{5!}{4!} = \frac{5 \times \cancel{4 \times 3 \times 2 \times 1}}{\cancel{4 \times 3 \times 2 \times 1}}$$

$= 5$

$$\frac{12!}{9!3!}$$

$$= \frac{12 \times 11 \times 10 \times \cancel{9 \times 8 \ldots\ldots}}{\cancel{9 \times 8 \ldots\ldots}\ 3 \times 2 \times 1}$$

$$= \frac{12 \times 11 \times 10}{3 \times 2 \times 1} = 220$$

Example 2: Write the following using factorial notation.

7 x 6 x 5 x 4 x 3 x 2 x 1

$= 7!$

$$\frac{18 \cdot 17 \cdot 16 \cdot 15}{3 \cdot 2 \cdot 1}$$

$$= \frac{18!}{14!\ 3!}$$

$$\frac{12 \cdot 11 \cdot 10}{4 \times 3 \times 2 \times 1}$$

$$= \frac{12!}{9!\,4!}$$

Pg 81 - #1, 3 (imp!), 4, 5

Example 3: Simplify

(A) $\dfrac{(n+1)!}{(n-1)!}$

$$= \frac{(n+1)(n)\cancel{(n-1)(n-2)\ldots\ldots}}{\cancel{(n-1)(n-2)(n-3)}\ldots\ldots\ldots}$$

$$= (n+1)(n) = \boxed{n^2+n}$$

(B) $\dfrac{n!}{(n-2)!} = \dfrac{n(n-1)\cancel{(n-2)!}}{\cancel{(n-2)!}}$

$$= n(n-1) = n^2 - n$$

Pg 81 - #1, 3 (imp!), 4, 5, 6

(C) $\dfrac{(n+3)(n+2)!}{(n+3)!}$

$$\left.\begin{matrix} 4\cdot 3! \\ 4\cdot 3\cdot 2\cdot 1 \end{matrix}\right\}$$

Example 4: Solve

(A) $$\frac{n!}{(n-2)!}=90$$

(B) $$\frac{(n+4)!}{(n+2)!}=6$$

Pg 82 #11 and worksheet

Section 2.3: Permutations When All Objects Are Distinguishable

${}_nP_r$ is the notation used to represent the number of permutations that can be made from a set of *n* different objects, where only *r* of them are used in each arrangement.

$${}_nP_r = \frac{n!}{(n-r)!}$$

Ex 1: Matt has downloaded 10 new songs from ITunes. He wants to create a playlist using 6 of these songs arranged in any order. How many different 6-song playlists can be created from his new downloaded songs?

$${}_{10}P_6 = \frac{10!}{(10-6)!} = \frac{10!}{4!} =$$

Ex 2: Determine all the possible 7-song playlists, then 8-song playlists, and finally 9-song playlists that Matt can create from 10 songs? How does the value of ${}_nP_r$ change as *r* gets closer to *n*?

${}_{10}P_7$ ${}_{10}P_9$

${}_{10}P_8$

Ex 3: There are 10 movies playing at Empire Cinemas. In how many ways can you see two of them consecutively?

Ex 4: **(A)** In how many ways can the letters of the word GRAPHITE be arranged?

$_8P_8$ $\underline{8}\ \underline{7}\ \underline{6}\ \underline{5}\ \underline{4}\ \underline{3}\ \underline{2}\ \underline{1}$

(B) How many three letter arrangements can be made from the word GRAPHITE?

$_8P_3$ $\underline{8}\ \underline{7}\ \underline{6}$

Solving a Permutation problem with Conditions

- Permutation problems sometimes involve conditions. In certain situations, objects may be arranged in a line where two or more objects must be placed together, or certain objects(s) must be placed in certain positions.

Ex 5: How many arrangements of the word FAMILY exist if the letters A and L must always be kept together?

$$\underbrace{\boxed{\_\ 2!\ \_}\ \_\ \_\ \_\ \_}_{5!}$$

$$\boxed{(2!)(5!)}$$

$${}_2P_2 \quad {}_5P_5$$

Ex 6: Find the number of permutations of the letters in the word KITCHEN if

(A) the letters K, C, and N must be together but not necessarily in that order.

$\boxed{3!\ \_\ \_\ \_}\ \underbrace{\_\ \_\ \_\ \_}_{5!}$

$3!\,5! = 720$

(B) the vowels must be kept together.

$\underbrace{\boxed{2!\ \_\ \_}\ \_\ \_\ \_\ \_\ \_}_{6!}$

$2!\,6! = 1440$

Ex 7: In how many ways can 3 girls and 4 boys be arranged in a row if no two people of the same gender can sit together?

$\underline{4}\ \underline{3}\ \underline{3}\ \underline{2}\ \underline{2}\ \underline{1}\ \underline{1} = 144 \quad \left\{ {}_4P_4 \cdot {}_3P_3 \right.$

Ex 8: Six actors and eight actresses are available for a play with four male roles and three female roles. How many different cast lists are possible?

$${}_6P_4 \cdot {}_8P_3 = \frac{6!}{(6-4)!} \cdot \frac{8!}{(8-3)!}$$

$$= \left(\frac{6!}{2!}\right)\left(\frac{8!}{5!}\right)$$

$$= (360)(336)$$

$$= 120\,960$$

Text: Pages 93-94: #s 1 b, d, f, 3, 4, 5, 7, 8, 10

<u>Solving a Permutation Problem involving Cases</u>

If a counting problem has one or more conditions that must be met,
*Consider each case that each condition creates, and
*Add the number of ways each case can occur to determine the total number of outcomes.

Ex (1): Tania needs to create a password for a social networking website.

- The password can use any digits 0 to 9 and/or any letters of the alphabet.
- The password is case sensitive, so she can use both lower and uppercase letters.
- A password must be at least 5 characters to a maximum of 7 characters.
- Each character must be used only once in the password.

How many different permutations are possible?

Find the total numbers of characters first: ____________________(This represents *n*)

Case 1: 5 character passwords for n = ___ and r = 5

Case 2: 6 character passwords for n = ___ and r = 6

Case 3: 7 character passwords for n = ___ and r = 7

Total number of passwords:

You Try

1. To open the garage door of Mary's house, she uses a keypad containing the digits 0 through 9. The password must be at least a 4 digit code to a maximum of 6 digits, and each digit can only be used once in the code. How many different codes are possible?

2. How many numbers can be made from the digits 2, 3, 4, and 5 if no digit can be repeated?
(Hint: consider 4 cases – four digit numbers, three digit numbers, two digit numbers, one digit numbers.)

Comparing arrangements created with and without repetition

Ex (2): A social insurance number (SIN) in Canada consists of a nine-digit number that uses the digits 0 to 9.

A) If there are no restrictions on the digits selected for each position in the number, how many SINs can be created if each digit can be repeated?

B) How does this compare with the number of SINs that can be created if no repetition is allowed?

Ex (3): Mrs. Pi has made up 12 multiple choice questions for a math quiz, where the answers A, B, C, and D are possible. How many different answer keys are possible if there are no restrictions?

#12,13,14 pg 94

Section 2.3 Cont'd....

Extension of Cases

Example 1: How many arrangements of the word QUEST are there if none of the vowels are together?

All possibities − vowels are together

$\underline{5}\ \underline{4}\ \underline{3}\ \underline{2}\ \underline{1} - \boxed{\underline{\ }\ 2!\ \underline{\ }}\ \underline{\ }\ \underline{4!}\ \underline{\ }$

$= 120 - 48 = \boxed{72}$

Example 2: How many arrangements of the word "active" are there if "c" and "e" are never together?

All − together

$\underline{6}\ \underline{5}\ \underline{4}\ \underline{3}\ \underline{2}\ \underline{1} - \boxed{\underline{\ }\ 2!\ \underline{\ }}\ \underline{\ }\ \underline{5!}\ \underline{\ }\ \underline{\ }$

$= 480$

Extension of Solving......

Solving Equations of the form $_nP_r = k$ $\frac{n!}{(n-r)!}$

Ex 1 Solve for n: $_nP_2 = 30$

$$\frac{n!}{(n-2)!} = 30$$

$$\frac{n(n-1)\cancel{(n-2)!}}{\cancel{(n-2)!}} = 30$$

$$n(n-1) = 30$$

$$n^2 - n = 30$$

$$n^2 - n - 30 = 0$$

$$(n-6)(n+5) = 0$$

$$\boxed{n=6} \text{ or } \cancel{n=-5}$$

Ex 2 Solve for n: $_{n-1}P_2 = 12$

$$\frac{(n-1)!}{(n-1-2)!} = 12$$

$$\frac{(n-1)!}{(n-3)!} = 12$$

$$\frac{(n-1)(n-2)\cancel{(n-3)!}}{\cancel{(n-3)!}} = 12$$

$$(n-1)(n-2) = 12$$

$$n^2 - 2n - n + 2 - 12 = 0$$

$$n^2 - 3n - 10 = 0$$

$$(n-5)(n+2) = 0$$

$$\boxed{n=5} \text{ or } \cancel{n=-2}$$

You try:

1. Solve for n: $_nP_2 = 56$
2. Solve for n: $_{n+1}P_2 = 20$

Section 2.4: Permutations when Objects are Identical

The number of permutations of *n* objects, where a are identical and b are identical and so on, is

$$P = \frac{n!}{a!b!}$$

Example 1: How many different 5-letter words can be formed from the word APPLE ?

Example 2: In how many ways can we rearrange the letters in the word MISSISSIPPI?

Example 3: In how many ways can the letters in the word BANANAS be arranged if the 1st letter must be A and the last letter is N?

textbook pg 104, #1, 2, 4, 5, 6, 7,9,10, 12,16**,17**

Section 2.5 - 2.7: Combinations

Definitions: Permutation vs. Combination

Example 1: Identify which situation is a permutation or which is a combination.

1. My fruit salad is a combination of apples, grapes and strawberries.

C.

2. The combination to the safe is 4-7-2

P

Example 2: In a lottery, six numbers from 1- 49 are selected (Lotto 6/49). A winning ticket must contan the same numbers but they may be in any order.

a) Determine the number of permutations. ${}_nP_r = \dfrac{n!}{(n-r)!}$

$${}_{49}P_6 = \frac{49!}{(49-6)!} = 10{,}068{,}347{,}520$$

b) Determine the number of combinations. ${}_nC_r = \dfrac{{}_nP_r}{r!} = \boxed{\dfrac{n!}{(n-r)!\,r!}}$

$${}_{49}C_6 = \frac{49!}{(49-6)!\,6!} = 13{,}983{,}816$$

Example 3: An assignment consists of three questions (A, B, C) and students are required to attempt two. How many possibilities exist?

$$ {}_3C_2 = \frac{3!}{(3-2)!\,2!} = 3 $$

Note: $ {}_3C_2 = \binom{3}{2} $

Types of Combination Problems

Type 1: basic Combinations - Selecting items and order is not important

Example 1: A committee of 4 people is to be formed from a group of 9 people. How many possible committees can be formed?

$${}_9C_4 = \binom{9}{4} = \frac{9!}{(9-4)!\,4!} = 126$$

Example 2: A pizza can have 3 toppings out of a possible 7. How many different pizza's can be made?

$${}_7C_3 = \binom{7}{3} = \frac{7!}{(7-3)!\,3!} = 35$$

Type 2: Combinations including Specific Items - Sometimes you will be forced to include or exclude particular items when making a combination.

Example 1: A school committee of 5 is to be formed from 12 students. how many committees can be formed if John must be on the committee?

$${}_{1}C_{1} \times {}_{11}C_{4} = 1 \times 330 = 330$$

[include workings!]

Example 2: From a deck of 52 cards, a 5 card hand is dealt. How many distinct 5 card hands are there if the queen of spades and the four of diamonds must be in the hand?

$${}_{1}C_{1} \times {}_{1}C_{1} \times {}_{50}C_{3}$$

$$= \frac{1!}{(1-1)!\,1!} \times \frac{1!}{(1-1)!\,1!} \times \frac{50!}{(50-3)!\,3!}$$

$$= 1 \times 1 \times 19600$$

$$= 19600$$

Type 3: Combinations from Multiple Selection Pools - when you have to select from multiple groups you have to multiple the separate classes together.

Example 1: A committee of 3 boys and 5 girls is to be formed from a group of 10 boys and 11 girls. how many committees are possible?

Example 2: From a deck of 52 cards, a 7 card hand is dealt. How many distinct hands are there if the hand must contain 2 spades and 3 diamonds?

Type 4: At Least/At Most - These questions will require you to ADD the possible cases together.

Example 1: A committee of 5 people is to be formed from a group of 4 men and 7 women. How many possible committees can be formed if at least 3 women are on the committee?

Example 2: From a deck of 52 cards, a 5 card hand is dealt. How many distinct hands can be formed if there are at least 2 queens?

Example 3: From a deck of 52 cards, a 7 card hand is dealt. How many distinct hands can be formed if there are at most 6 black cards?

Simplify:

$$\left(\frac{n}{n-1}\right)$$

$$\left(\frac{n+1}{n-1}\right)$$