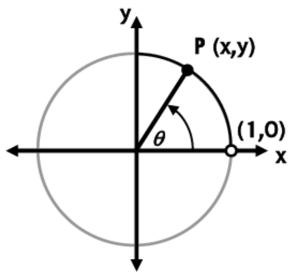
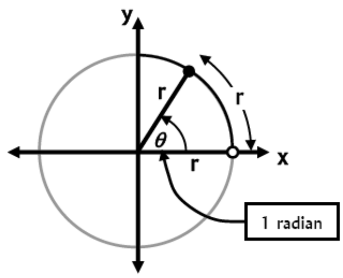
Chapter 4 – Trigonometry & the Unit Circle

4.1 Angles & Angle Measure

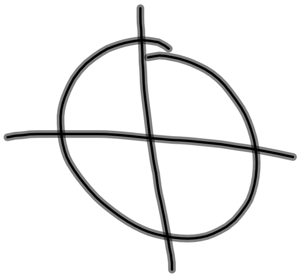
Thus far in your mathematical explorations, all of the angles have been measured using degrees. By measuring in degrees (θ), we can indicate the position of a point P as it rotates from its initial location (1,0) about the origin.



There is, however, another well known method of measuring angles known as ***radians***. When measuring an angle using radians, we measure the resulting arc length on the circle's circumference. Formally defined, one ***radian*** is the measure of the angle formed when the arc length of a circle has the same length as the radius of the circle.



By converntion, angles measured on **clockwise direction** are negative ~~**positive**~~ while those in a **counterclockwise** are said to be ~~**negative**~~ positive.



$360° = 2\pi\ radians$

$180° = \pi\ radians$

$90° = \frac{\pi}{2} radians$

$45° = \frac{\pi}{4}\ radians$

****Note: Angles measures without units are considered to be radians

How many degrees are in 1 radian?

- $\dfrac{360^\circ}{2\pi} = \dfrac{2\pi \text{ radians}}{2\pi}$
- $\dfrac{180}{\pi} = 1 \text{ radian}$

$$1 \text{ radian} = \frac{180}{\pi} \text{ degrees}$$

$$1 \text{ degree} = \frac{\pi}{180} \text{ radians}$$

These two formulae can now be used to convert angular measure into degrees or radians.

To convert from radians to degrees:	To convert from degrees to radians:
⇨ radians $\times \dfrac{180}{\pi}$	⇨ degrees $\times \dfrac{\pi}{180}$

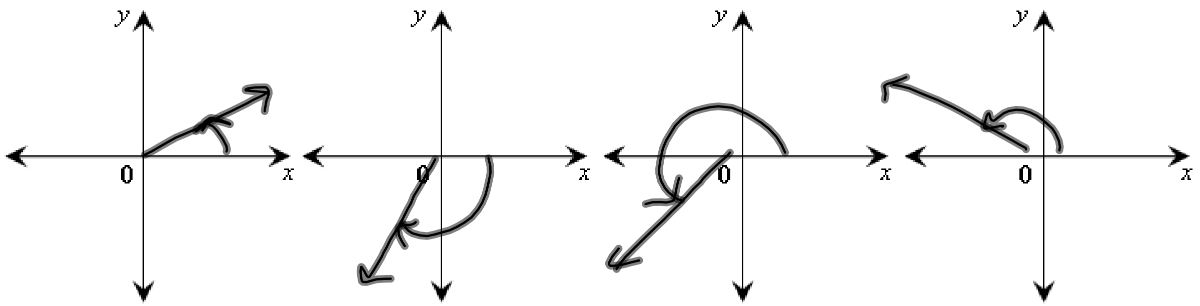
Example 1

Convert Between Degree and Radian Measure

Draw each angle in standard position. Change each degree measure to radian measure and each radian measure to degree measure. Give answers as both exact and approximate measures (if necessary) to the nearest hundredth of a unit.

a) $30°$ **b)** $-120°$

c) $\frac{5\pi}{4}$ **d)** 2.57



$= 30° \times \frac{\pi}{180}$

$= \frac{\pi}{6}$

$\doteq 0.52$

$-120 \times \frac{\pi}{180}$

$= -\frac{2\pi}{3}$

$\doteq -2.09$

$\frac{5\pi}{4} \times \frac{180}{\pi}$

$= 225°$

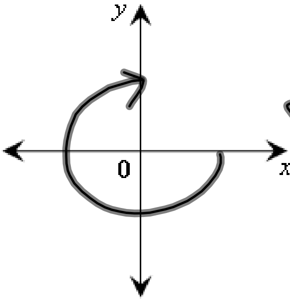
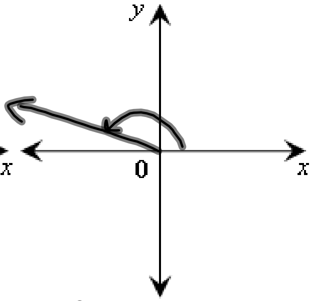
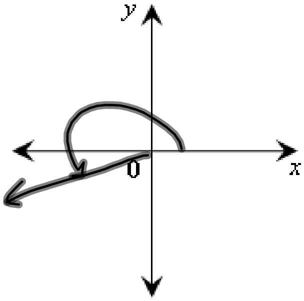
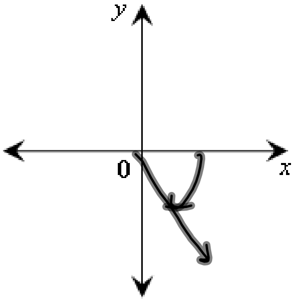
$2.57 \times \frac{180}{\pi}$

$\doteq 147°$

Your Turn

Draw each angle in standard position. Change each degree measure to radian measure and each radian measure to degree measure. Give answers as both exact and approximate measures (if necessary) to the nearest hundredth of a unit.

-270° 150° $\frac{7\pi}{6}$ -1.2

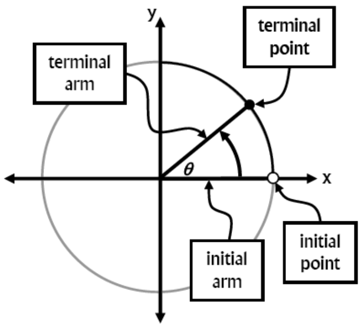





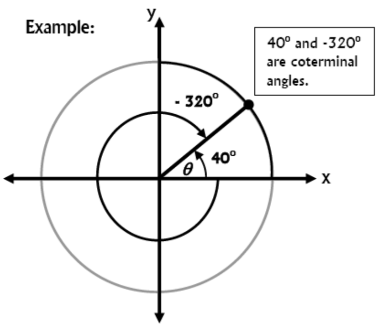
$-270^\circ \times \frac{\pi}{180} = -\frac{3\pi}{2}$

$150^\circ \times \frac{\pi}{180} = \frac{5\pi}{6}$

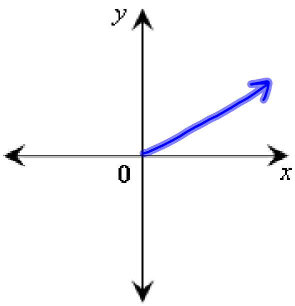
$\frac{7\pi}{6} \times \frac{180}{\pi} = 210^\circ$

$-1.2 \times \frac{180}{\pi} = -68.75$





a) $40°$



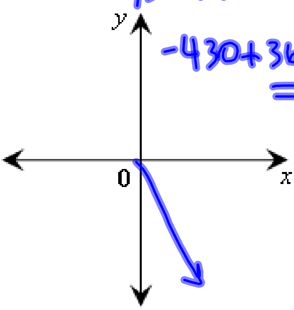
$40° + 360° = 400°$

$40 - 360° = -320°$

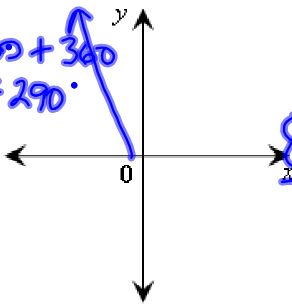
b) $-430°$

$-430 + 360° = -70°$

$-430 + 360 + 360$
$= 290°$



c) $\frac{8\pi}{3}$



$\frac{8\pi}{3} + 2\pi = \frac{14\pi}{3}$

$\frac{8\pi}{3} - 2\pi = \frac{2\pi}{3}$

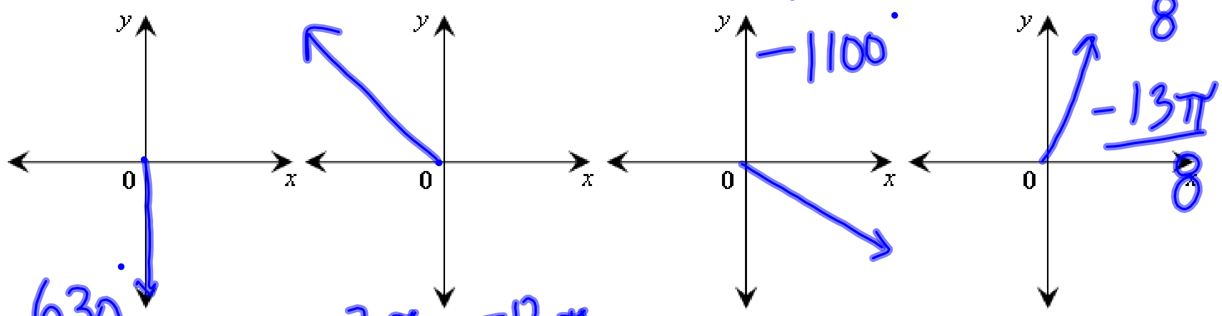
$\frac{2\pi}{3} - 2\pi = \frac{-4\pi}{3}$

a) $270°$

b) $\frac{-5\pi}{4}$ -2π

c) $-740°$ $340°$ $-1100°$

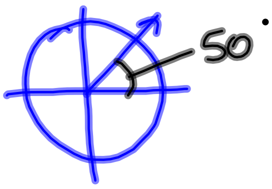
d) $\frac{3\pi}{8}$ $\frac{19\pi}{8}$ $\frac{-13\pi}{8}$



$630°$
$-90°$

$\frac{3\pi}{4}$ $\frac{-13\pi}{4}$

Coterminal Angles



$50°, 410°, 770°, 1130°, \ldots\ldots$

$\theta = 50° \pm 360n, n \in N$

$50° \times \frac{\pi}{180°}$

$\frac{5\pi}{18}$

$\theta = \frac{5\pi}{18} \pm 2\pi n, n \in N$

Example 3 (pg 172): Express the angles coterminal with 110 degrees in general form. Identify the angles that satisfy the domain $-720^0 \leq x < 720^0$

$$\theta = 110^\circ \pm 360^\circ n, n \in N$$

$$470^\circ, -250^\circ, -610^\circ$$

Ex 3B: Express the angles coterminal with $\frac{8\pi}{3}$ in general form. Identify the angles that satisfy the domain $-4\pi \leq \theta < 4\pi$

$$\theta = \frac{8\pi}{3} \pm 2\pi n, n \in N$$

$$\cancel{\frac{8\pi}{3} + 2\pi = \frac{14\pi}{3}}$$

$$\frac{8\pi}{3} - 2\pi = \frac{2\pi}{3}$$

$$-\frac{4\pi}{3} \qquad -\frac{10\pi}{3}$$

All arcs that subtend a right angle ($\frac{\pi}{2}$) have the same central angle, but they have different arc lengths depending on the radius of the circle. The arc length is proportional to the radius. This is true for any central angle and related arc length.

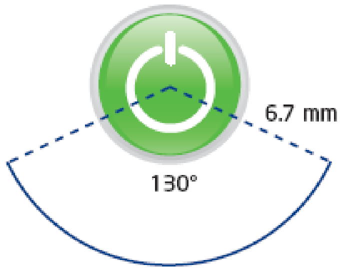
Consider the circle with radius r and the sector with central angle θ .

This formula, a = θ r, works for any circle, where θ is in radians.

arc length of a circle: $a = \theta r$

Example 4:

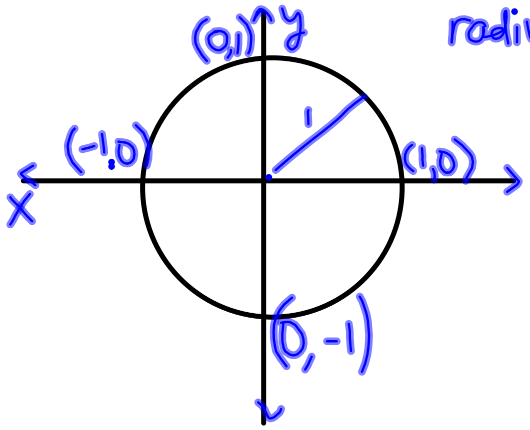
Rosemarie is taking a course in industrial engineering. For an assignment, she is designing the interface of a DVD player. In her plan, she includes a decorative arc below the on/off button. The arc has central angle 130° in a circle with radius 6.7 mm. Determine the length of the arc, to the nearest tenth of a mm.



4.2 The Unit Circle

Unit Circle

A circle with centre at the origin (0,0) and a radius of 1 unit is referred to as a unit circle.



radius 1

Example 1

Equation of a Circle Centred at the Origin

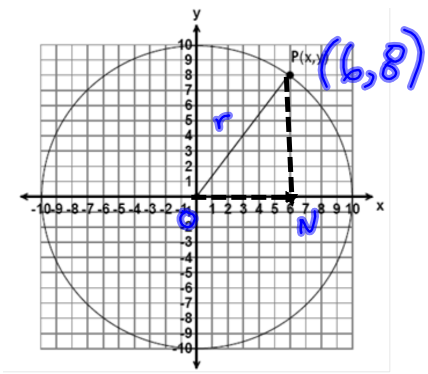
(a) Identify the radius of the circle.

(b) Identify the coordinates of the point P.

(6, 8)

(c) Draw a vertical line segment PN from P to the x-axis.

(d) Draw a horizontal line segment NO from N to the origin.



(e) What type of triangle have you created?

right

(f) Determine the lengths of PN and NO.

6 8

(g) Write an equation that relates the three sides of the triangle.

$6^2 + 8^2 = r^2$

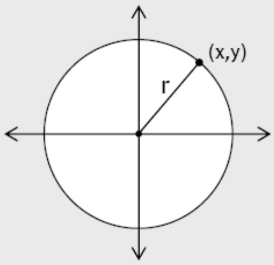
(h) Write an equation of any circle that is centered at the origin and has a radius of ‘r’.

$x^2 + y^2 = r^2$

<u>EQUATION OF A CIRCLE CENTERED AT THE ORIGIN</u>

Any circle with its center located at the origin O(0,0) and radius ‘r’ is defined by the equation:

$$\boxed{x^2 + y^2 = r^2}$$



Your Turn

a) radius = 6

$x^2+y^2=6^2$

$x^2+y^2=36$

b) radius = $\sqrt{10}$

$x^2+y^2=(\sqrt{10})^2$

$x^2+y^2=10$

c) diameter = 6

$3=r$

$x^2+y^2=9$

d) passing through the point (-5, 12)

$(-5)^2+12^2=r^2$

$25+144=r^2$

$169=r^2$

$r=13$

$\boxed{x^2+y^2=169}$

Example 2

Determine Coordinates for Points of the Unit Circle

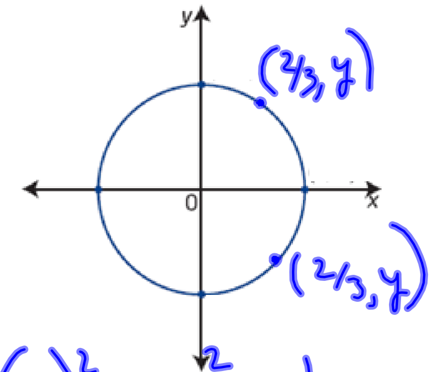
Determine the coordinates for all points on the unit circle that satisfy the conditions given. Draw a diagram in each case.

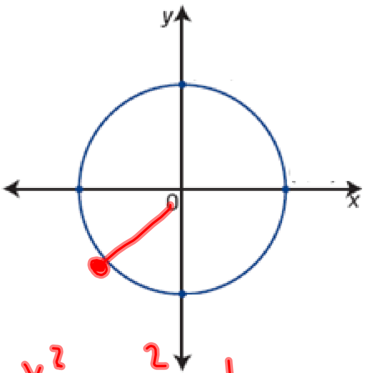
a) the x-coordinate is $\frac{2}{3}$

b) the y-coordinate is $-\frac{1}{\sqrt{2}}$ and the point is in quadrant III

Solution

a) Coordinates on the unit circle satisfy the equation $x^2 + y^2 = 1$.





$$\left(\frac{2}{3}\right)^2 + y^2 = 1$$
$$\frac{4}{9} + y^2 = 1$$
$$y^2 = 1 - 4/9$$
$$y^2 = \frac{5}{9}$$
$$y = \pm\sqrt{\frac{5}{9}}$$
$$y = \pm\frac{\sqrt{5}}{3}$$
$$\left(\frac{2}{3}, \frac{\sqrt{5}}{3}\right) \text{ \& } \left(2/3, -\frac{\sqrt{5}}{3}\right)$$

$$x^2 + y^2 = 1$$
$$x^2 + \left(\frac{-1}{\sqrt{2}}\right)^2 = 1$$
$$x^2 + \frac{1}{2} = 1$$
$$x^2 = \frac{1}{2}$$
$$x = \pm\frac{1}{\sqrt{2}}$$
$$\left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$$

rewrite

$$\frac{-1}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{-\sqrt{2}}{2}$$
$$\left(\frac{-\sqrt{2}}{2}, \frac{-\sqrt{2}}{2}\right)$$

Determine the missing coordinate(s) for all points on the unit circle satisfying the given conditions.
Use the circle diagram and determine which quadrant(s) the points lie in.

$\left(-\frac{5}{8}, y\right)$

$x^2 + y^2 = 1$

$\left(-\frac{5}{8}\right)^2 + y^2 = 1$

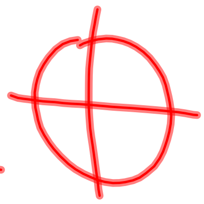
$\frac{25}{64} + y^2 = 1$

$y^2 = 1 - \frac{25}{64}$

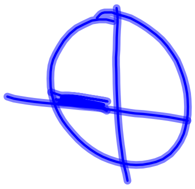
$y^2 = \frac{39}{64}$

$y = \pm \frac{\sqrt{39}}{8}$

(Q2, Q3)



$\left(x, \frac{5}{13}\right)$ where the point is in quadrant II



$x^2 + \left(\frac{5}{13}\right)^2 = 1$

$x^2 = 1 - \frac{25}{169}$

$x^2 = \frac{144}{169}$

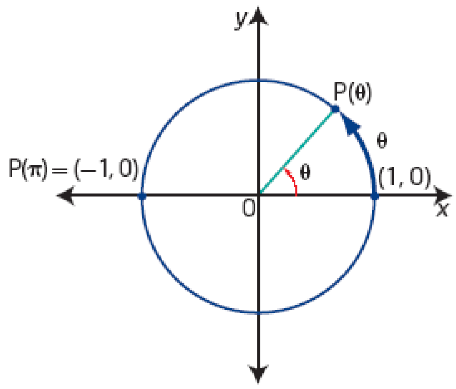
$x = \pm \frac{12}{13}$

$x = -\frac{12}{13}$

Relating Arc Length and Angle Measure in Radians

The form

in radians; and r is the radius, applies to any circle, as long as a and r are measured in the same units. In the unit circle, the formula becomes a = θ(1) or a = θ. This means that a central angle and its subtended arc on the unit circle have the same numerical value.



$P(2\pi) = (x, y)$

$P(\pi/6) = (?, ?)$

You can use the function P(θ) = (x, y) to link the arc length, θ, of a central angle in the unit circle to the coordinates, (x, y), of the point of intersection of the terminal arm and the unit circle.

For example, if θ = π, the point is (-1, 0). Thus, you can write P(π) = (-1, 0).

Example 3

Multiples of $\frac{\pi}{3}$ on the Unit Circle

a) On a diagram of the unit circle, show the integral multiples of $\frac{\pi}{3}$ in the interval $0 \le \theta \le 2\pi$.

b) What are the coordinates for each point $P(\theta)$ in part a)?

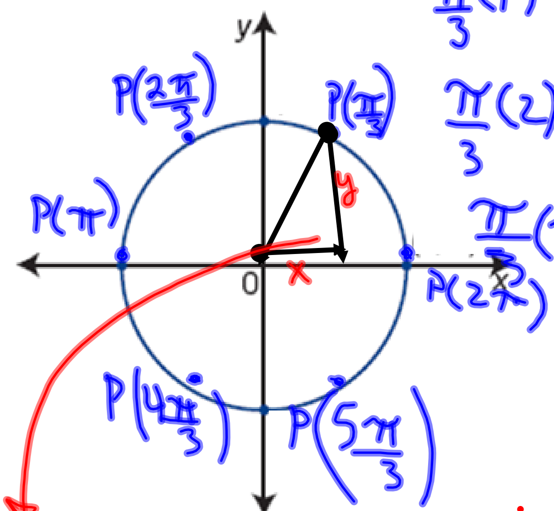
c) Identify any patterns you see in the coordinates of the points.

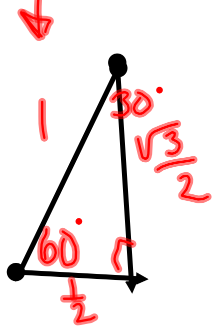
Solution

$\frac{\pi}{3}(1) = \frac{\pi}{3}$ $\frac{\pi}{3}(4) = \frac{4\pi}{3}$

$\frac{\pi}{3}(2) = \frac{2\pi}{3}$ $\frac{\pi}{3}(5) = \frac{5\pi}{3}$

$\frac{\pi}{3}(3) = \pi$ $\frac{\pi}{3}(6) = 2\pi$

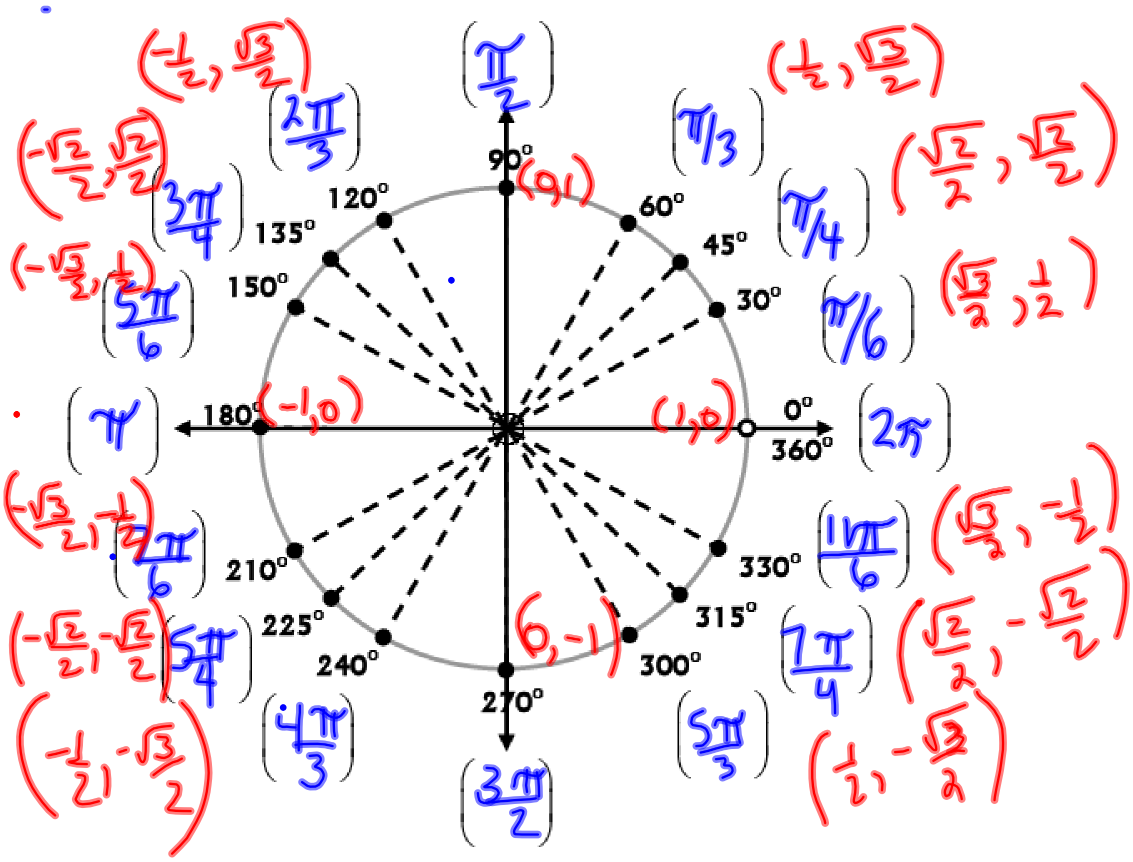




$P(\pi/3) = \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ $P\left(\frac{4\pi}{3}\right) = \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

$P(2\pi/3) = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ $P\left(\frac{5\pi}{3}\right) = \left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$

$P(\pi) = (-1, 0)$ $P(2\pi) = (1, 0)$


90°
120°
135°
150°
180°
210°
225°
240°
270°
300°
315°
330°
0°
360°
30°
45°
60°
(0,1)
(-1,0)
(1,0)
(0,-1)
$\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
$\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
$\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$
$\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$
$\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
$\left(-\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$
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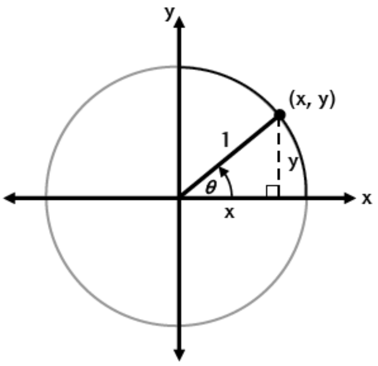
$\left(\frac{\sqrt{3}}{2}, -\frac{1}{2}\right)$
$\left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$
$\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$
$\frac{\pi}{2}$
$\pi/3$
$\pi/4$
$\pi/6$
2π
$\frac{2\pi}{3}$
$\frac{3\pi}{4}$
$\frac{5\pi}{6}$
π
$\frac{7\pi}{6}$
$\frac{5\pi}{4}$
$\frac{4\pi}{3}$
$\frac{3\pi}{2}$
$\frac{5\pi}{3}$
$\frac{7\pi}{4}$
$\frac{11\pi}{6}$

4.3 Trigonometric Ratios

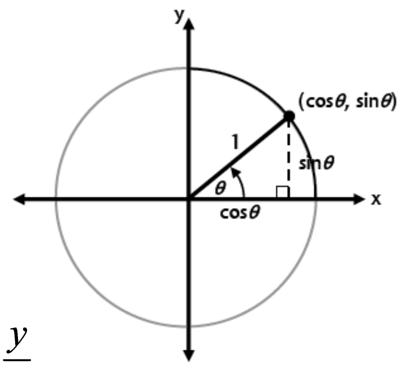


Using the SOHCAHTOA rule we have the following ratios:

$$\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}} \Rightarrow \cos\theta = \frac{x}{1}$$
$$\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}} \Rightarrow \sin\theta = \frac{y}{1}$$

Thus we have the equality:

$$x = \cos\theta$$
$$y = \sin\theta$$



$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{y}{x}$$

<u>Reciprocal Trigonometric Ratios</u>

Cosecant

$\csc\theta = \frac{1}{\sin\theta}$

Secant

$\sec\theta = \frac{1}{\cos\theta}$

Cotangent

$\cot\theta = \frac{1}{\tan\theta}$

Note: if $\sin\theta = -\frac{\sqrt{3}}{2}$, $\csc\theta = \frac{2}{-\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = -\frac{2\sqrt{3}}{3}$

Example 1

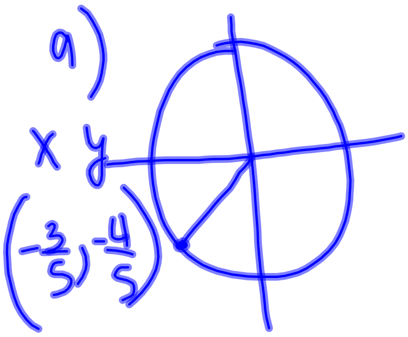
Determine the Trigonometric Ratios for Angles in the Unit Circle

The point $A\left(-\frac{3}{5}, -\frac{4}{5}\right)$ lies at the intersection of the unit circle and the terminal arm of an angle θ in standard position.

a) Draw a diagram to model the situation.

b) Determine the values of the six trigonometric ratios for θ. Express answers in lowest terms.

a)



b) $\cos\theta = x = \frac{-3}{5}$

$\sin\theta = y = -\frac{4}{5}$

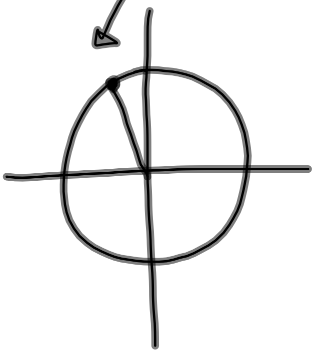
$\tan\theta = \frac{y}{x} = \frac{-\frac{4}{5}}{-\frac{3}{5}} = 4/3$

$\sec\theta = \frac{1}{\cos\theta} = \frac{5}{-3}$

$\csc\theta = \frac{1}{\sin\theta} = \frac{5}{-4}$

$\cot\theta = \frac{1}{\tan\theta} = \frac{3}{4}$

$B\left(\frac{-1}{3}, \frac{2\sqrt{2}}{3}\right)$



$\sin\theta = y$

$= \frac{2\sqrt{2}}{3}$

$\csc\theta = \frac{3\sqrt{2}}{4}$

$\cos\theta = \frac{-1}{3}$

$\sec\theta = -3$

$\cot\theta = \frac{1}{-2\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}}$

$= \frac{\sqrt{2}}{-4}$

$\tan\theta = \frac{\frac{2\sqrt{2}}{3}}{\frac{-1}{3}}$

$= -2\sqrt{2}$

Example 2

Exact Values for Trigonometric Ratios

Determine the exact value for each. Draw diagrams to illustrate your answers.

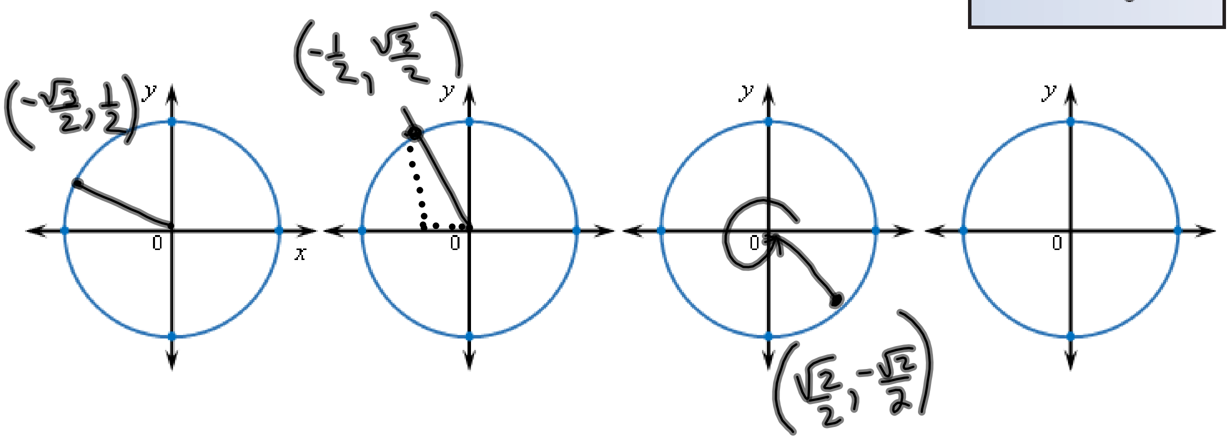
a) $\cos \frac{5\pi}{6} = -\frac{\sqrt{3}}{2}$ **b)** $\sin\left(-\frac{4\pi}{3}\right) = \frac{\sqrt{3}}{2}$

c) $\sec 315°$ ↓ cos **d)** $\cot 270°$

Did You Know?

By convention, if the domain is given in radian measure, express answers in radians. If the domain is expressed using degrees, give the answers in degrees.

Solution



$$\sec 315° = \frac{1}{\cos 315°}$$

$$= \frac{2}{\sqrt{2}} \cdot \frac{\sqrt{2}}{\sqrt{2}} = \frac{2\sqrt{2}}{2} = \sqrt{2}$$

$\tan\frac{\pi}{2}$ $\csc\frac{7\pi}{6}$ $\sin(-300^\circ)$ $\sec 60^\circ$