Chapter 11: Permutations, Combinations and the Binomial Theorem

Factorial - !

n! = n (n-1) (n-2) (n-3)........(3)(2)(1)

Example: 5! = 5 x 4 x 3 x 2 x 1

Example 1: Simplify

① $\frac{6!\,3}{(5-2)!} = \frac{6!\,3}{3!} = \frac{[6 \times 5 \times 4 \times \cancel{3 \times 2 \times 1}](3)}{\cancel{3 \times 2 \times 1}}$

$= 360$

② $\frac{(n+1)!}{(n-1)!} = \frac{(n+1)(n)\cancel{(n-1)(n-2)}\ldots}{\cancel{(n-1)(n-2)}\ldots}$

$= n^2 + n$

Note:

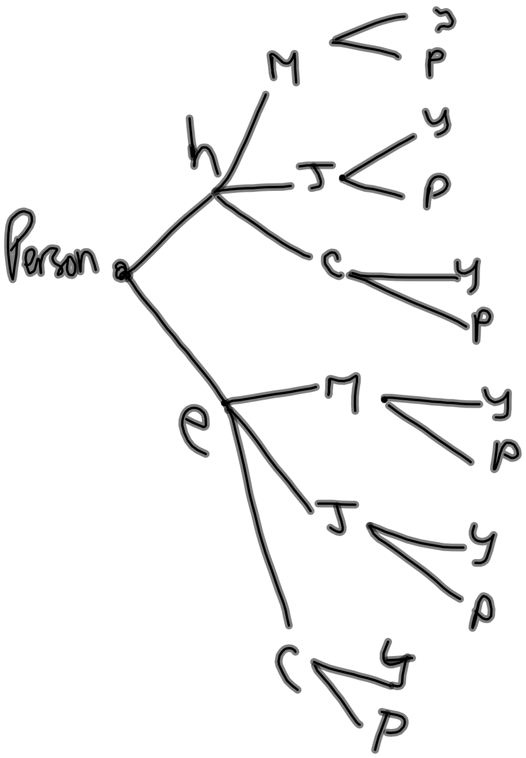
- you can't take the factorial of a negative number
- $1! = 1$
- $0! = 1$

Fundamental Counting Principle: helps determine the total # of possible arrangements that can occur within a group(s).

Example 1: A cafe has a lunch special consisting of one from each category.

- ham sandwich, egg sandwich
- milk, juice, coffee
- yogurt, pie

In how many ways can you chose your lunch?



$\boxed{\text{12 Combos}}$

Note:

- if we have x options in the 1st event, y options in the second event, and z options in the 3rd event, then

total # of possible combinations = (x)(y)(z)

- pay attention to whether repetition is allowed!

Example 2: A computer store sells 6 computers, 4 monitors, 5 printers and 3 multi-media packages. Assuming all items are different, how many computer packages are available?

$6 \times 4 \times 5 \times 3$

$= 360$

Example 3: How many possible arrangements are there in the 649 lottery?

$\underline{49}\ \underline{48}\ \underline{47}\ \underline{46}\ \underline{45}\ \underline{44}$ (6 #'s to choose from 1-49, no repetition)

1.0068×10^{10}

Example 4: A license plate consists of three letters followed by three digits. Determine the total # of possible license plate #'s that can be printed.

$\underline{26}\ \underline{26}\ \underline{26}\quad \underline{10}\ \underline{10}\ \underline{10}$

17,576,000

Example 5: Using the digits 1-5, how many positive 3 digit #'s can be formed if:
a) there are no restrictions?

$\underline{5}\ \underline{5}\ \underline{5}$ 125

b) the number formed is odd and repetition is allowed?

$\underline{5}\ \underline{5}\ \underline{3} = 75$

c) there are no repeats allowed?

$\underline{5}\ \underline{4}\ \underline{3}$ 60

d) there are no repeats allowed and the # is odd?

$\underline{4}\ \underline{3}\ \underline{3}$ 36

~~3 4 3~~

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#1, 3, 4, 10

Permutations

- order of items in the groupings matter

i.e. ABC is not the same as ACB

4 Possible Cases

- all items in group are used and all items are different
- all items in group are used but some items are repeated
- only some items in the group are used
- permutations that may have repeats and constraints

Example 1: In how many ways can we rearrange the letters in the word MATH?

$\underline{4}\ \underline{3}\ \underline{2}\ \underline{1} = 24$

Ex 2: A lock has a 3# combination that will open it. the #'s go from 1 - 50. How many different combinations are there?

$\underline{50}\ \underline{50}\ \underline{50} = 125\ 000$

Permutations with repeats in the original group

Total # of arrangements = $\dfrac{n!}{a!b!c!}$ where a, b, c are repeated objects

Example 1: In how many different ways can we rearrange the letters in the word FUNCTION?

$\dfrac{8!}{2!} = 8 \times 7 \times 6 \times 5 \times 4 \times 3$

$= 20160$

Example 2: In how many ways can we rearrange the letters in the word MISSISSIPPI?

$$\frac{11!}{4!\,4!\,2!} = 34650$$

Example 3: In how many ways can 3 cars (Red, green & blue) be parked in 5 parking stalls?

R G B E E

$$\frac{5!}{2!} = 60$$

Example 4: An electrical panel has 5 switches. In how many ways can the switches be positioned if 3 must be up and 2 must be down?

$$\frac{5!}{3!\,2!} = 10$$

Example 5: In how many ways can the letters in the word BANANAS be arranged if the 1st 2 letters must be As and the 3rd letter is not allowed to be an A?

fixed ∴ $\underbrace{\underline{A}\ \underline{A}}$ $\underline{4}\ \underline{4}\ \underline{3}\ \underline{2}\ \underline{1}$

No repetition for A's.

$$\frac{96}{2!} = 48$$

Example 6: A tower is made from 4 red, 5 yellow and 3 blue lego blocks.
a) In how many ways can these lego blocks be arranged so as to create a different tower?

$$\frac{12!}{4!5!3!} = 27,720$$

b) In how many different ways can the blocks be arranged if the top of the tower has 4 red blocks?

$\underline{R}\ \underline{R}\ \underline{R}\ \underline{R}\ \underline{8}\ \underline{7}\ \underline{6}\ \underline{5}\ \underline{4}\ \underline{3}\ \underline{2}\ \underline{1}$

fixed

$$\frac{8!}{5!3!} = 56$$

Permutation Formula

- can be used when asked to determine how many different arrangements there are for a subset of an original group.

$$\text{total \# objects} \leftarrow {}_{n}P_{R} = \frac{n!}{(n-r)!}$$

$R \downarrow$ # in Subset

Example 1: In how many ways can we pick a secretary, president and vice president from a group of 8 people?

$${}_{n}P_{r} = {}_{8}P_{3} = \frac{8!}{(8-3)!} = \frac{8!}{5!} = 8 \times 7 \times 6 = 336$$

pg 524-527 # 2, 5, 6, 7, 9, 11, 12, 13, 14, 16, 17, 19, 20, 22, 25, 26, 27

pg 534-536 #1-11, 13-15, 17-21,23, C1, C3

Combinations:

order of items in the groupings do not matter

i.e. BAC is the same as ABC

$${}_{n}C_{r} = \frac{n!}{(n-r)!\,r!} \qquad\qquad \binom{n}{r}$$

Ex 1: Mrs W is going to make up a quiz on chapter 8 from a pool of 10 questions. She is going to put 6 questions on the quiz. How many different quizzes can she make up?

$${}_{10}C_{6} = \frac{10!}{4!\,6!} = 210$$

Ex 2: The social justice group is made up of 4 level 1's, 5 level 2's and 6 level 3's.

a) How many ways can a committee of 5 students be selected if there are no restrictions?

$${}_{15}C_5 = 3003$$

(workings)

b) How many ways can a committee of 5 include exactly 2 level 3's?

$${}_6C_2 \times C = 1260$$

level 3's

c) How many committees of students have at most two level 3's?

exactly 2 level 3's $\rightarrow$ ${}_6C_2 \cdot {}_9C_3 = 1260$

exactly 1 $\rightarrow$ ${}_6C_1 \cdot {}_9C_4 = 756$

exactly 0 $\rightarrow$ ${}_6C_0 \cdot {}_9C_5 = 126$

$\boxed{\text{Total} = 2142}$

Be careful with word problems that include the words

exactly, at least, at the most

Example 6: Coast Radio has 6 tickets for an upcoming concert. 12 men and 8 women have qualified to win the tickets.
a) In how many ways can 6 winners be selected?

$${}_{20}C_{6} = \frac{20!}{14!\,6!} = 38760$$

b) In how many ways can 6 tickets be awarded if at least one winner is a female?

"at least" one would mean that 1, 2, 3, 4, 5 or 6 of the winners could be female.

Best approach - Find the number of possibilities that NONE are female (all male) and then subtract from all possibilities.

No Females

$${}_{12}C_{6} = 924$$

$$\text{total} = 38760 - 924$$
$$= 37836$$

c) In how many ways can the 6 tickets be awarded if at least 3 are men?

exactly 3 men $_{12}C_3 \times {}_8C_3 = 12320$

exactly 4 men $_{12}C_4 \times {}_8C_2 = 13860$

- exactly 5 men $_{12}C_5 \times {}_8C_1 = 6336$

exactly 6 men $_{12}C_6 \times {}_8C_0 = 924$

total 33440

Section 11.3: Binomial Theorem

$$(a+b)^0 = 1$$

$$(a+b)^1 = a+b$$

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$$

$$(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

$$(a+b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$$

...

Pascal's Triangle:

1

1 1

1 2 1

1 3 3

1 4 6 4 1

1 5 10 10 5 1

$$\binom{0}{0}$$

$$\binom{1}{0} \quad \binom{1}{1}$$

$$\binom{2}{0} \quad \binom{2}{1} \quad \binom{2}{2}$$

$$\binom{3}{0} \quad \binom{3}{1} \quad \binom{3}{2} \quad \binom{3}{3}$$

$$\binom{4}{0} \quad \binom{4}{1} \quad \binom{4}{2} \quad \binom{4}{3} \quad \binom{4}{4}$$

${}_0C_0$

${}_1C_0$, ${}_1C_1$

Example 1: Expand $(p + q)^6$

$$= 1p^6q^0 + 6p^5q^1 + 15p^4q^2 + 20p^3q^3 + 15p^2q^4 + 6p^1q^5 + 1p^0q^6$$

$$\Rightarrow p^6 + 6p^5q + 15p^4q^2 + 20p^3q^3 + 15p^2q^4 + 6pq^5 + q^6$$

Example 2: a) Expand $(2a - 3b)^4$

$= 1(2a)^4(-3b)^0 + 4(2a)^3(-3b)^1 + 6(2a)^2(-3b)^2 + 4(2a)(-3b)^3 + (-3b)^4$

$= 16a^4 - 96a^3b + 216a^2b^2 - 216ab^3 + 81b^4$

b) What is the third term in the expansion of $(4b - 5)^6$?

$15(4b)^4(-5)^2$

$= 96000b^4$

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#1, 2, 3, 4, 5, 6, 7ab